Chapter 1

Introduction

1.1 Definition and Terminology

Definition 1.1.1. Differential equation (DE.) is an equation involving the derivative of a unknown function. For example,

$$\frac{dy}{dx} = 3x$$

or

$$y' - 2xy = 2.$$

Classification by type

If an equation contains one independent variable, it is said to be an **ordinary differential equation (ODE)**. Examples are

$$y' = x, \quad y'' + 2xy = 1,$$

or

$$xy'' = y \sin x, \quad y' = \cos xy + e^x$$

The position of a particle of mass m moving along a straight line is denoted by $u(t)$, the force acting to it is $F(t, u(t), u'(t))$, then

$$mu''(t) = F(t, u(t), u'(t)). \tag{1.1}$$

On the other hand, if an equation contains more than one independent variable, it is said to be a **partial differential equation** (PDE). Examples are the potential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

or the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0.$$

Classification by order

$y', y'', \cdots, y^{(n)}$ are called first order, second order, $\cdots$, n-th order derivative. The highest such number is called the **order of a differential equation**.

(1) 1-st order DE.

$$y' = x + y$$
$$(1 + x^2)y' + y^2 = 1$$
$$(y')^2 = y^2 + 1.$$

(2) 2-nd order DE.

$$y'' + 3y' + 2y = 0$$
$$y(y'')^2 + ky + 1 = 0.$$

(3) n-th order DE.

$$\boxed{y^{(n)} + p_1(x)y^{(n-1)} + \cdots p_n(x)y = r(x)\ .} \tag{1.2}$$

Usually an ODE is given in the following implicit form:

$$F(x, y, y', \cdots, y^{(n)}) = 0. \tag{1.3}$$

Under certain condition, we can solve it for the highest order term $y^{(n)}$:

$$\boxed{y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})\ .} \tag{1.4}$$

This is called a **normal(standard) form**.

Classification by linearity

If the equation is linear in y, y', y'', etc., then we say it is **linear differential equation**. Otherwise, it is a **nonlinear differential equation**. Examples of linear differential equations are

$$y'' = -y, \quad x^2y' = xy - 1, \quad \frac{dy}{dx} = F(x)y.$$

Examples of non-linear differential equations are

$$(y')^2 + 1 = y, \quad yy' = x + 1 \quad y' = x + \cos(y) + 1.$$

A first order linear differential equation is sometimes written as a **differential form**

$$M(x, y)dx + N(x, y)dy = 0.$$

Example 1.1.2 (Differential form of a 1st order ODE)**.**

$$(x^2 + 1)y\frac{dy}{dx} + y - \sin x = 0$$

Multiplying it by dx we obtain the differential form

$$(y - \sin x)dx + (x^2 + 1)ydy = 0$$

Solution

Definition 1.1.3. A function ϕ defined on an interval I and having at least n derivatives which are continuous on I is called a **solution** of the differential equation if it satisfies a given differential equation.

For example, ϕ is a solution of the ODE $F(x, y, y', \cdots, y^{(n)}) = 0$ if

$$F(x, \phi(x), \phi'(x), \cdots, \phi^{(n)}(x)) = 0.$$

Example 1.1.4 (Families of solution)**.** The function $y = c_1e^{-x} + x - 1$ is the solution of ODE

$$y' + y = x.$$

It is called a **one-parameter family of solutions**. A solution free of parameter is called a **particular solution**. $y = x - 1$ is the particular solution of DE $y' + y = x$.

The function $y = c_1e^x + c_2xe^x + x^2 - 2$ is the solution of ODE

$$y'' - 2y' + y = 4x$$

for any constants c_1, c_2. These are **two parameter family of solutions**.

Interval of definition

Or domain of definition.

Definition 1.1.5. Implicit solution, Explicit solution of the differential equation. Any function given by the relation of the form $G(x, y) = 0$ is an implicit function.

Example 1.1.6 (Implicit solutions)**.** Consider the ODE $xdx + ydy = 0$. The function $x^2 + y^2 = 25$ is an implicit solution.

Singular Solutions

Example 1.1.7 (Piecewise defined solutions)**.** Consider ODE $xy' = 4y$. $y = cx^4$ is a one-parameter family of solutions in the interval $(-\infty, \infty)$ but you may consider a piecewise defined solution

$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geq 0 \end{cases}$$

This solution cannot be obtained by a single choice of parameter in $y = cx^4$. This kind of solution is called a singular solution.

System of Differential Equations

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned} \tag{1.5}$$

1.2 Initial value problem

1.2.1 IVP-first order, second order, ...

Recall the **one parameter family of solutions**. A DE must have appropriate condition so that it can have unique solution. For example

Solve

$$xy' + y = 1 \text{ subject to } y(1) = 1$$

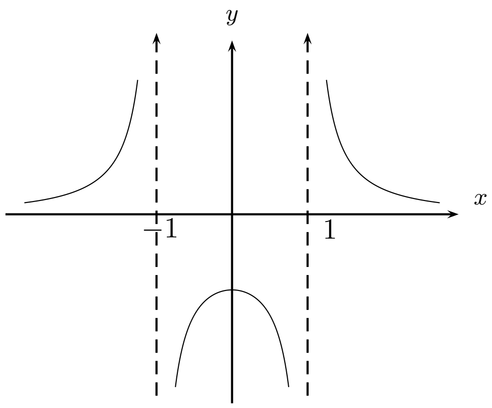
or

$$a(x)y'' + b(x)y' + c(s)y = d(x), \text{ with IC. } y(0) = 1, y'(0) = 2.$$

Usually the number of conditions equals the order of DE.

Example 1.2.1 (Interval of definition of solutions)**.** You can verify $y = 1/(x^2+c)$ is a solution of DE $y' + 2xy^2 = 0$. With I.C. $y(0) = -1$, you get $y = 1/(x^2 - 1)$. Three cases arises:

- As a function, $y = 1/(x^2 - 1)$ is a function defined on the real line except $x = -1, 1$.
- As a solution of ODE $y' + 2xy^2 = 0$, it can take any interval in $(-\infty, -1)$, $(-1, 1)$ or $(1, \infty)$ as the domain.
- As a solution of the IVP $y' + 2xy^2 = 0$, $y(0) = -1$, the interval should be taken an interval in $(-1, 1)$.



1.2.2 First order linear equation

We consider the following.

$$y' + p(x)y = q(x) \tag{1.6}$$

Multiplying the integrating factor $\mu(x)$ to get

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{1.7}$$

Assume the following:

$$\frac{d}{dx}(\mu(x)y) = \mu(x)y' + \mu'(x)y. \tag{1.8}$$

Comparing with (2.8) we have

$$\mu'(x) = \mu(x)p(x) \text{ or } \frac{\mu'(x)}{\mu(x)} = p(x). \tag{1.9}$$

Integrating this, we obtain

$$\boxed{\mu(x) = \exp\left(\int^x p(t)dt\right)}. \tag{1.10}$$

Subst. this into (2.9). Then by (2.8) we have

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x).$$

Thus integrating we obtain

$$\mu(x)y = \int^x \mu(t)q(t)dt + C.$$

Now the solution is

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(t)q(t)dt + C\right]$$

or

$$\boxed{y(x) = \exp\left(-\int^x p(t)dt\right)\left[\int^x e^{\int^\xi p(t)dt}q(\xi)d\xi + C\right]}$$

Example 1.2.2. Solve IVP

$$\begin{aligned} y' + 2xy &= x \\ y(0) &= 0. \end{aligned}$$

Sol. We find the integrating factor and multiply it

$$\mu(x) = e^{\int 2xdx} = e^{x^2}$$

to get

$$e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}.$$

$$(e^{x^2}y)' = xe^{x^2}$$

$$e^{x^2}y = \int^x te^{t^2}dt + C = \frac{1}{2}e^{x^2} + C,$$

$$y = \frac{1}{2} + Ce^{-x^2}$$

Use IC. to get

$$y = \frac{1}{2}(1 - e^{-x^2}).$$

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1.2.3 Existence and Uniqueness

Example 1.2.3. [IVP can have several solutions] Find the solution of

$$y' = 2y^{\frac{1}{2}} \quad (1.11)$$
$$y(0) = 0. \quad (1.12)$$

Sol. By inspection, we find the following two solutions easily:

$$y(x) = 0, \quad y(x) = x^2.$$

Also, for any $x_0 > 0$, the function

$$y = \begin{cases} 0, & 0 \le x < x_0 \\ (x - x_0)^2, & x \ge x_0 \end{cases}$$

is a solution. □

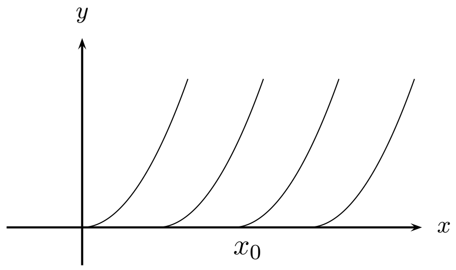


Figure 1.1: Non unique solutions

Things to consider

(1) Existence

(2) Uniqueness

(3) Valid interval

Theorem 1.2.4. *Existence and uniqueness If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on $D = (a, b) \times (c, d)$ and if $(x_0, y_0) \in D$, then the solution of IVP*

$$y' = f(x, y) \quad (1.13)$$
$$y(x_0) = y_0 \quad (1.14)$$

exists in a nhd $x_0 - h < x < x_0 + h$ *and the solution is unique.*

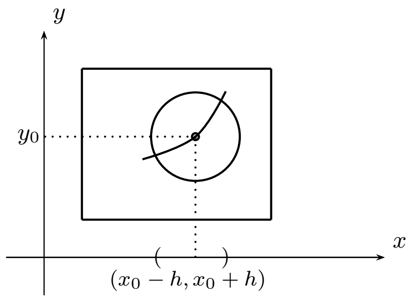


Figure 1.2: Interval of existence

in the Example 1.2.3: $f(x,y) = 2y^{\frac{1}{2}}$. Hence $\frac{\partial f}{\partial y} = y^{-\frac{1}{2}}$ does not satisfy the condition of the theorem.

Example 1.2.5. Find the solution of DE:

$$y' = xy^2$$
$$y(0) = 1.$$

Sol. $y = \frac{1}{1-x^2/2}$ is a solution. Since $f(x,y) = xy^2$ and $\frac{\partial f}{\partial y} = 2xy$ are continuous near $x = 0$, $y = 1$, this is the only solution. The interval of validity is $-\sqrt{2} < x < \sqrt{2}$.

□

Example 1.2.6. Solve

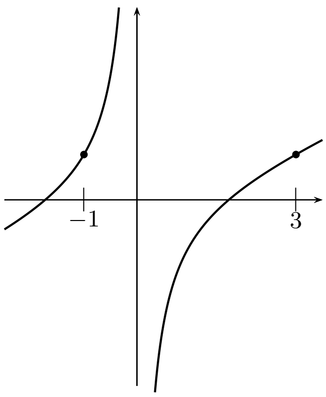
$$\begin{aligned} y' + \frac{y}{x-1} &= 1 \\ y(x_0) &= y_0. \end{aligned}$$

On $(-100, 1)$ or on $(1, 10^{10})$, $p(x) = 1/(x-1)$ is continuous, so it has a unique solution there. Here $y = \frac{x-1}{2} + \frac{C}{x-1}$.

Example 1.2.7. Solve IVP and determine the interval where solution is valid.

$$\begin{aligned} xy' + y &= 2x, \\ y(3) &= 2. \end{aligned}$$

Sol. Divide by x. Then $y' + \frac{1}{x}y = 2$ and $\mu(x) = e^{\ln x} = x$.



interval of $y = x - \frac{3}{x}$

Thus we obtain

$$(xy)' = 2x$$
$$xy = x^2 + C$$
$$y = x + \frac{C}{x}.$$

It is valid on $(0, \infty)$ or $(-\infty, 0)$. Since $x \neq 0$, $x > 0$. With IC. $y = x - \frac{3}{x}$.

Example 1.2.8. Solve the above problem with IC. changed to $y(-1) = 2$.

Sol. With the new IC, we see $y = x - \frac{3}{x}$ is the solution. But the valid interval is now $(-\infty, 0)$.

Exercise 1.2.9. (1) Determine order of the following DE. Are they linear non-linear ?

(a) $x^2y'' + xy' + y = 2$
(b) $x(y')^2 + \sin x + y = 2$
(c) $y'' + y = \sec x$
(d) $y'' + \cos(x + y) = e^x$

(2) Find a solution of the form $y = e^{rx}$ in the following.

(a) $y'' + 4y = 0$
(b) $y'' + 5y' + 4y = 0$
(c) $y'' + 2y' - 3y = 0$

(d) $y''' - 4y'' + 3y' = 0$

(3) Determine order of the following DE. Are they linear nonlinear ?

(a) $u_{xx} + ku_{yy} = 0$

(b) $uu_x = u_y$

(c) $c^2 u_{xx} = u_t$

(d) $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$

(4) Find a solution of the following.

(a) $y' + 2y = x$

(b) $y' + 2xy = e^{-x^2}$

(c) $y' + y = xe^x$

(d) $y' - \frac{1}{x}y = xe^x$

(e) $y' - 2y = e^{2x}$

(f) $xy' + 2y = \sin x$

(g) $y' = \frac{1}{y}e^{x-y}$

(h) $y' + \frac{1}{x}y = \cos x$

(5) Solve

(a) $xy' + 3y = x - 1, \quad y(1) = 1$

(b) $(1 - x)y' + y = 1 - x, \quad y(0) = 1$

(c) $x^2 y' - xy - 1 + \frac{3}{2}x = 0, \quad y(1) = 1$

(d) $y' + \frac{1}{x}y = \frac{\sin x}{x}, \quad y(\frac{\pi}{2}) = 1$

1.3 Mathematical Model

Many phenomena in the nature of engineering is expressed in terms of DE. The processes are called **(mathematical modelling)**

Example 1.3.1. Velocity and acceleration

The position of a car running on a straight road is denoted by $y(t)$(t- time) The velocity is $\Delta y/\Delta t$. In general the instant velocity is $v(t) = \lim_{\Delta t \to 0} \Delta y/\Delta t = dy/dt$. If velocity is given as $10t(m/sec)$ what is distance traveled for 5 min?

Sol. We have

$$y' = 10t$$

So

$$y = 5t^2 + C$$

with $y(0) = 0$ the distance traveled 5 min is $5 \cdot 5^2 m = 125m$. Acceleration is $a = dv/dt = d^2y/dt^2$. Here $10(m^2/sec)$.

□

Example 1.3.2. (**Free fall**)
The position of a falling object of mass m is denoted by $y(t)$, velocity v, with force $f(t, y, y')$, we have by Newton's law

$$my''(t) = f(t, y, y')$$

Free fall: gravitational force g ignoring air resist we have $v' = y''$

$$my'' = mg \tag{1.15}$$

If $g = 9.8m/sec^2$ what is the distance of falling for 20 sec?

Sol.

$$y'' = 9.8$$

$$y' = 9.8t + C$$

Since $v(0) = 0$

$$y' = 9.8t$$

Thus

$$y = 4.9t^2 + C$$

$y(0) = 0$

$$y = 4.9t^2$$

For 20 seconds $4.9 \times (20)^2 = 1,960m$.

□

If the friction is considered we have $y'' = g - k(y')^2$. Consider the fall of parachute. It is experimentally known that the force by air resistance proportional to the square of velocity. Hence we get by Newton's second law

$$mv' = mg - bv^2,$$

b constant depending on the Parachute. Since $y' = v$, we obtain

$$y'' = g - \frac{b}{m}(y')^2. \tag{1.16}$$

Example 1.3.3. [Population dynamics, Decay of isotope] The mass of an isotope was 5 gram at a moment. After 3 min, it became 4 gram. Then let $y(t)$ be the remaining mass at t min. Write an equation of $y(t)$. What is half life ?(The time that takes to reduce to half of the original amount.)

The decay rate $\frac{dy}{dt}$ is proportional to the current amount.

$$\frac{dy}{dt} = -ky \quad (k > 0).$$

Here k is some constant dependent on the material. The solution is $y(t) = Ce^{-kt}$.

Example 1.3.4. [Newton's law of cooling]

$$\frac{dT}{dt} = k(T - T_m).$$

Example 1.3.5. [Spread of disease] Let $x(t)$ be the number of people who have contracted the disease(say a flu) and $y(t)$ be the number of people who have not exposed. The rate of disease spread is proportional to xy because people interact. So

$$\frac{dx}{dt} = kxy.$$

Or if one infected person is introduced into a small community of n people, then $x + y = n + 1$ so

$$\frac{dx}{dt} = kx(n + 1 - x).$$

Example 1.3.6. [Chemical Reaction] The molecules of a substance A decompose into smaller molecules. The rate is prop. to the amount of first substance not decomposed.

$$\frac{dX}{dt} = kX$$

or

$$\frac{dX}{dt} = k(a - X)(b - X)$$

Example 1.3.7. [Draining a Tank] Torricelli's law: The speed of outflux of water through a sharp edged hole at the bottom of a tank filled to a depth of h is the same as the speed of the body would require in the free fall from the height h: Thus,

$$\frac{dV}{dt} = -A_h\sqrt{2gh},$$

where A_h is the area of hole and h the height of water.

Example 1.3.8. [**Electric circuit**]:

For a circuit with R (resistor) and L (inductor) only, the Kirchhoff law says **sum of voltage drop across the inductor** $L(di/dt)$ **and the voltage drop across the resistor** iR is the same as the impressed voltage $E(t)$. So the system is described by

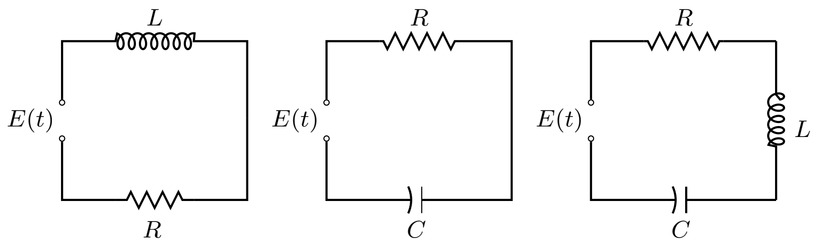
$$L\frac{di}{dt} + Ri = E(t).$$

Likewise, since the **voltage drop across the capacitor with capacitance** C **is** $q(t)/C$, the circuit with R - C only is described by

$$Ri + \frac{1}{C}q = E(t). \tag{1.17}$$

The current and charge q are related by $i = dq/dt$, we have

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{1.18}$$



RL-circuit RC-circuit RLC-circuit

If the charge of the condenser is $q(t)$, input voltage is $E(t)$, then $L\frac{di}{dt} = Lq''$ is added, so

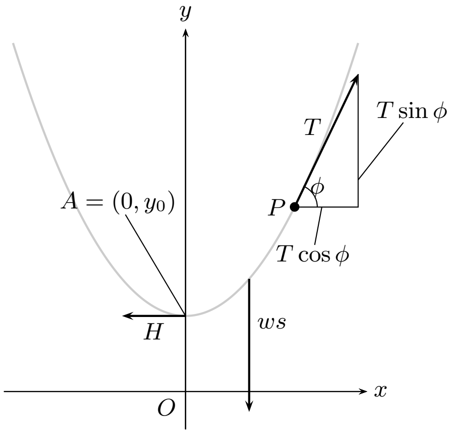
$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E(t).$$

Example 1.3.9. [**Hanging Cable**]: Let s be the length of the cable between y axis to the point P, then the weight of the cable of the portion is ws and from the relations

$$H = T\cos\phi, \quad ws = T\sin\phi, \quad \frac{dy}{dx} = \frac{ws}{T}.$$

Since $\frac{ds}{dx} = \sqrt{1 + (\frac{dy}{dx})^2}$, we get

$$\frac{d^2y}{dx^2} = \frac{w\sqrt{1 + (\frac{dy}{dx})^2}}{T}.$$



This is a nonlinear second order equation.

Chapter 2

First order DE

2.1 Solution curves

2.1.1 Direction Fields and Integral Curves

Given a DE. $y' = f(x, y)$, a collection of direction vectors with slope $f(x, y)$ placed at (x, y) is called a **direction field** of the DE.

It is sometimes hard to find the solutions of a DE. It is helpful to draw the tangent curves to the family of curves to determine the shape of solution.

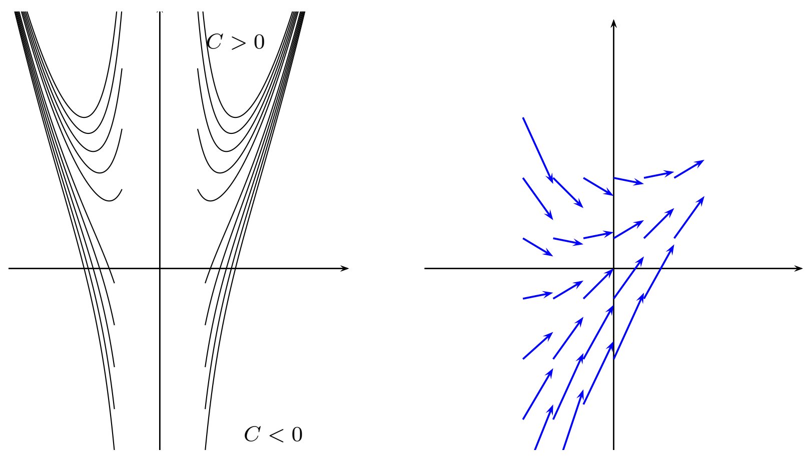


Figure 2.1: Integral curves $y = x^2 + \frac{C}{x^2}$ and direction fields of $y = x + Ce^{-x}$

The (many) solutions of a DE. are called **integral curves** of the DE.

Example 2.1.1. The DE

$$y' + y = x + 1$$

has a general solution $y = x + Ce^{-x}$. Integral curves are given by left picture of Figure 2.1.

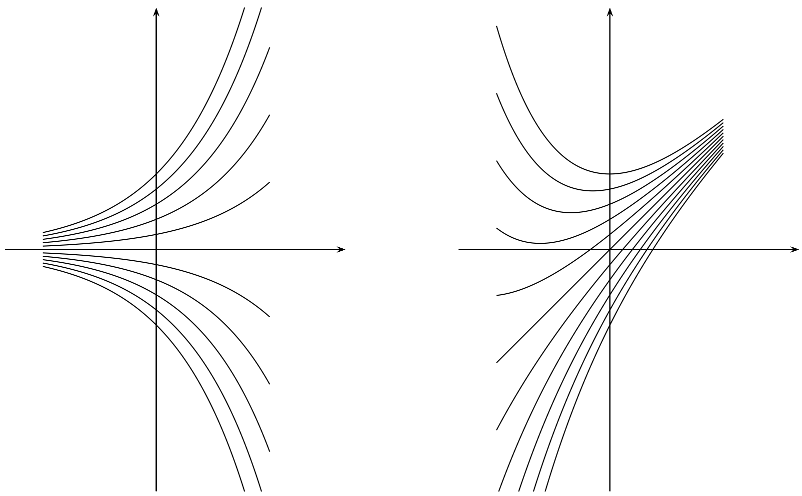


Figure 2.2: Integral curves of $y = Ce^x$, $y = x + Ce^{-x}$

Example 2.1.2. Integral curves of

$$y' = \frac{x^2}{1-y}$$

are $y - \frac{y^2}{2} + C = \frac{x^3}{2}$.

Example 2.1.3. The solution of $xy' + 2y = 4x^2$ is $y = x^2 + \frac{C}{x^2}$ and its integral curves are as in left side of Figure 2.1.

Example 2.1.4. Draw direction fields of

(1) $y' + y = x + 1$

(2) $xy' + 2y = x^2 + 1$

(3) $y' + y^2 = x + 1$

Sol. (1) right side of figure 2.1.

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2.1.2 Autonomous First order DEs

A DE. with no independent variable is called **autonomous**. An autonomous first order DE can be give as $F(y', y) = 0$ or in the followng forms

$$\frac{dy}{dx} = f(y), \tag{2.1}$$

$$\text{or } \frac{dx}{dt} = k(n + 1 - x), \quad \frac{dT}{dt} = k(T - T_m).$$

2.1.3 Critical points

The zeros of $f(y)$ in (2.1) is very important and called a **critical point, stationary point, equilibrium point**.

Example 2.1.5. The D.E. $\frac{dP}{dt} = P(a - bP)$ has equilibrium points $P = 0$ and $P = a/b$. The behavior of solution near the Critical points are given in the table.

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	minus	dec.	Down
$(0, a/b)$	Plus	Inc.	Up
$(a/b, \infty)$	minus	dec.	Down

Table 1.

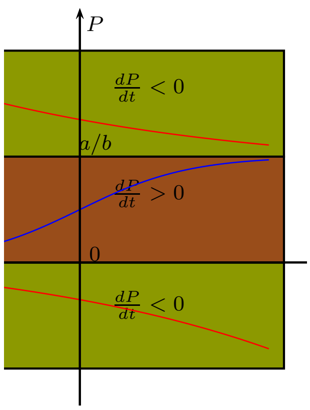

Figure 2.3: Phase plane

Example 2.1.6. The autonomous D.E.

$$\frac{dy}{dx} = (y-1)^2$$

has single critical point 1. The solution is given by $y = 1 - 1/(x+c)$. The solutions with IC's $y(0) = -1$ (resp. $y(0) = 2$), are given by

$$y = 1 - \frac{1}{x + 1/2} \text{ on } -\frac{1}{2} < x < \infty \text{ (resp. } y = 1 - \frac{1}{x-1} \text{ on } -\infty < x < 1).$$

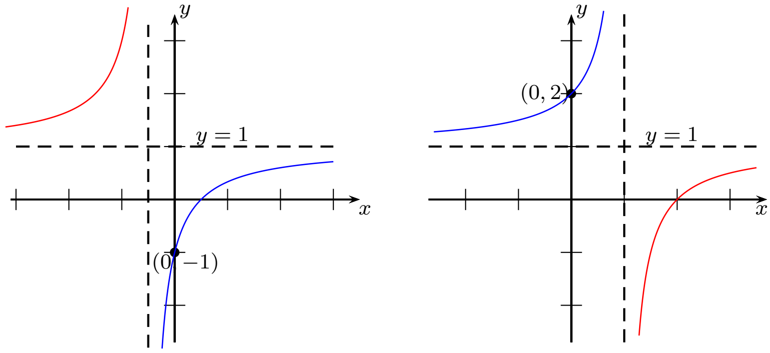

Figure 2.4: Sol. with distinct I.C's

Attractors and Repellers

Assume $y(x)$ is a non constant solution of an autonomous DE and c is a critical point of the DE. Basically four possibilities: As $x \to \pm\infty$

(1) $\lim_{x\to\pm\infty} y = c$ (point y moves towards c — attractor) -asymptotically stable.

(2) point y moves away from c — repeller - unstable.

(3) point y moves towards c one side, and moves away from the other side — neither attractor nor repeller - semi stable.

(4) point y moves towards c one side, and moves away from the other side — neither attractor nor repeller - semi stable.

In the above example, $y = 1$ is semi stable.

Translation property

If $y(x)$ is a solution of auto DE $dy/dx = f(y)$, then $y(x-k)$ is also a solution for any constant k.

2.2 Separable Equations

If a DE $y' = f(x, y)$ can be written in the form

$$\boxed{\frac{dy}{dx} = \frac{g(x)}{h(y)}},$$

it is said to be **variable separable**. We can set

$$g(x)dx = h(y)dy.$$

Integration gives

$$\int^x g(x)\,dx = \int^y h(y)\frac{dy}{dx}dx + C.$$

Example 2.2.1. Solve

$$\frac{dy}{dx} = \frac{\sin x}{1+y^2}.$$

Multiply $(1+y^2)dx$ to get

$$(1+y^2)dy = \sin x\,dx.$$

Integrating, we obtain

$$y + \frac{y^3}{3} = -\cos x + C.$$

Example 2.2.2. Solve the IVP

$$\frac{dy}{dx} = \frac{3x^2+2x}{2y+2}, \quad y(0) = 1.$$

Sol. Multiplying by $2y + 2$ we obtain

$$(3x^2 + 2x) - (2y + 2)\frac{dy}{dx} = 0.$$

Integrate

$$\int^x (3x^2 + 2x)dx - \int^y (2y + 2)dy = 0.$$

$$x^3 + x^2 - (y^2 + 2y) = C$$

or

$$y + 1 = \pm\sqrt{x^3 + x^2 + C_1}.$$

Using the IC., we obtain $y = -1 + \sqrt{x^3 + x^2 + 4}$.

Losing a solution

Example 2.2.3. Solve

$$\frac{dy}{dx} = y^2 - 4.$$

Sol. Dividing by $y^2 - 4$, we obtain

$$\frac{dy}{y^2 - 4} = dx \; or \; \frac{1}{4}\left[\frac{1}{y-2} - \frac{1}{y+2}\right] dy = dx.$$

Integrate

$$\ln\left|\frac{y-2}{y+2}\right| = 4x + c.$$

Thus

$$y = 2\frac{1 + ce^{4x}}{1 - ce^{4x}}.$$

From the RHS of the DE., we see $y = \pm 2$ are trivial equilibrium solution. One of them $y = 2$ can be obtained as setting $c = 0$. But the other one $y = -2$ cannot be obtained, we have lost it during the division. (This solution corresponds to the case $x \to \infty$).

To determine the constant C to get an unique solution we need some condition, called **initial condition (IC)**. A DE. with IC. is called **initial value problem (IVP)**.

Example 2.2.4 (IVP)**.** Solve

$$\cos x(e^{2y} - y)\frac{dy}{dx} = e^y \sin 2x, \text{ I.C. } y(0) = 0.$$

Sol. Dividing by $e^y \cos x$, we get

$$\frac{e^{2y} - y}{e^y}dy = \frac{\sin 2x}{\cos x}dx.$$

$$\int (e^y - ye^{-y})dy = 2\int \sin x\, dx$$

and hence

$$e^y + ye^{-y} + e^{-y} = -2\cos x + C.$$

Using the I.C. we can get $C = 4$.

Exercise 2.2.5. (1) Solve DE.

(a) $y' + y^2 \cos x = 0$
(b) $(y + e^y)y' = x - e^x$
(c) $xy' = (1 - y^2)^{1/2}$
(d) $y' = \frac{x}{1+2y}$
(e) $y' = \frac{x^2}{y^2-4}$
(f) $y' = x^3(1 + y^2)$
(g) $yy' = \sin x$
(h) $\frac{dr}{d\theta} = r \sin\theta$
(i) $(x \ln x)y' = y^2$
(j) $xy' = y^3 + y^2$
(k) $y' \cos y = 1$
(l) $dr = r \tan\theta d\theta$

2.3 Linear Equations

2.3.1 IVP-first order, second order, ...

Examples of linear DE: $xy' + y = 1,\ \ y(1) = 1$.

Example 2.3.1. First order linear differential equation:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)\ .$$

Dividing by $a_1(x)$ we obtain a normal(standard) form:

$$\boxed{\frac{dy}{dx} + P(x)y = f(x)}. \tag{2.2}$$

Second order linear differential equation:

$$a(x)y'' + b(x)y' + c(x)y = d(x), y(0) = 1, y'(0) = 2.$$

Homogeneous DE

If $f \equiv 0$ in (2.2), we say it is **Homogeneous.**

Example 2.3.2. Find the general solution of

$$\frac{dy}{dx} + P(x)y = 0. \tag{2.3}$$

Sol. Multiply a function $\mu(x)$ to both sides of (2.3) to get

$$\boxed{\mu(x)y' + P(x)\mu(x)y = 0}.$$

Assume lhs is the derivative of $\mu(x)y$. Then we see

$$\frac{d}{dx}[\mu(x)y] = \mu(x)y' + \mu'(x)y = \mu(x)y' + P(x)\mu(x)y \tag{2.4}$$

and hence we have $\frac{\mu'(x)}{\mu(x)} = P(x)$, i.e.,

$$\mu(x) = e^{\int P(x)dx}. \tag{2.5}$$

Here $\mu(x)$is called an **integrating factor** substituting into (2.4), we see

$$e^{\int P(x)dx}y' + P(x)e^{\int P(x)dx}y = 0.$$

Thus

$$\frac{d}{dx}(ye^{\int P(x)dx}) = 0$$

and we obtain $ye^{\int P(x)dx} = C$. Thus

$$y = Ce^{-\int P(x)dx}.$$

□

Nonhomogeneous DE

Consider solving the following nonhomogeneous equation:

$$\boxed{y' + p(x)y = q(x)}. \tag{2.6}$$

We note that the solution consists of two parts: $y = y_c + y_p$, where y_c is a solution of the homogeneous DE:

$$\boxed{y' + p(x)y = 0} \tag{2.7}$$

and y_p is any particular solution of (2.6). In fact,

$$(y_c + y_p)' + p(x)(y_c + y_p) = y_c' + p(x)y_c + [y_p' + p(x)y_p] = q(x).$$

The solution of the homogeneous DE is known previously as

$$y_c = \exp^{-\int p(x)dx}.$$

Variation of parameters

Now we want to find a particular solution of (2.6) by a procedure called **variation of parameters**. Let $y_p = u(x)y_1(x)$, where $y_1(x)$ any solution of the homogeneous DE (2.7). Substitute it into (2.6) gives

$$u\frac{dy_1}{dx} + y_1\frac{du}{dx} + p(x)uy_1 = q(x), \ \textit{or}\ u\left[\frac{dy_1}{dx} + p(x)y_1\right] + y_1\frac{du}{dx} = q(x),$$

from which we get

$$y_1(x)\frac{du}{dx} = q(x).$$

$$\frac{du}{dx} = \frac{q(x)}{y_1(x)}.$$

Integrating,

$$u(x) = \int \frac{du}{dx}dx = \int \frac{q(x)}{y_1(x)}dx.$$

So

$$y_p = \exp^{-\int p(x)dx} \int \exp^{\int p(x)dx} q(x)dx.$$

Integrating factor

We introduce another method. They are actually quite similar. Multiplying the integrating factor $\mu(x)$ we get

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{2.8}$$

Assume the following:

$$\frac{d}{dx}(\mu(x)y) = \mu(x)y' + \mu'(x)y. \tag{2.9}$$

Comparing with (2.8) we have

$$\mu'(x) = \mu(x)p(x) \text{ or } \frac{\mu'(x)}{\mu(x)} = p(x). \tag{2.10}$$

Integrating this we have

$$\boxed{\mu(x) = \exp\left(\int^x p(t)dt\right)}. \tag{2.11}$$

Subst. into (2.9). Then by (2.8) we have

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x).$$

Thus integrating

$$\mu(x)y = \int^x \mu(t)q(t)dt + C$$

and solution is

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(t)q(t)dt + C\right]$$

or

$$\boxed{y(x) = \exp\left(-\int^x p(t)dt\right)\left[\int^x e^{\int^\xi p(t)dt}q(\xi)d\xi + C\right]}.$$

Caution: We note that this formula is valid only when the DE is given of the form (2.6), i.e., the coeff. of leading term y' is 1.

Example 2.3.3. Solve IVP

$$y' + 2y = 1 \tag{2.12}$$

$$y(0) = 1. \tag{2.13}$$

Sol. Multiply $\mu(x) = e^{2x}$ we have $\frac{d}{dx}(e^{2x}y) = e^{2x}$ and we see

$$e^{2x}y = \frac{1}{2}e^{2x} + C.$$

$$y = \frac{1}{2} + Ce^{-2x}.$$

IC. $y(0) = 1$. Then $C = \frac{1}{2}$ and so the solution is $y = \frac{1}{2} + \frac{1}{2}e^{-2x}$.

$\square$

Example 2.3.4. Solve IVP

$$\begin{aligned} y' + 2xy &= x \\ y(0) &= 0. \end{aligned}$$

Sol. Integrating factor is

$$\mu(x) = e^{\int 2x dx} = e^{x^2}.$$

$$e^{x^2} y' + 2xe^{x^2} y = xe^{x^2}$$
$$(e^{x^2} y)' = xe^{x^2}.$$

$$e^{x^2} y = \int^x te^{t^2} dt + C = \frac{1}{2} e^{x^2} + C,$$
$$y = \frac{1}{2} + Ce^{-x^2}.$$

Use IC. to get

$$y = \frac{1}{2}(1 - e^{-x^2}).$$

If IC. is changed to $y(0) = \frac{1}{2}$ then $C = 0$ and solution is $y = \frac{1}{2}$.

Piecewise linear DE.- Discontinuous forcing term

Example 2.3.5. Solve IVP

$$y' + y = f(x), \quad y(0) = 0, \; where \; f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & x > 1. \end{cases}$$

Sol. I.F. is e^x. Multiplying it we get

$$e^x y' + e^x y = e^x f(x), \; or \; (e^x y)' = e^x f(x).$$

Integrating,

$$e^x y(x) = \int_0^x e^x f(x) dx + C_1.$$

If $x \leq 1$, then

$$e^x y(x) = e^x + C_1.$$

With IC., we see $C_1 = -1$.

$$y(x) = 1 - e^{-x}.$$

If $x > 1$,

$$e^x y(x) = \int_0^1 e^x dx + C_2 = e^1 - 1 + C_2,$$

so

$$y(x) = (e^1 - 1 + C_2)e^{-x}.$$

Using the continuity at $x = 1$, we get $C_2 = 0$. Hence

$$y(x) = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1 \\ (e-1)e^{-x}, & x > 1. \end{cases}$$

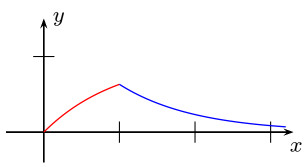

Figure 2.5: Sol. with discontinuous forcing

The error function

There are some important functions that are defined through integrals. Examples are

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad and \; erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \tag{2.14}$$

We know that $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}/2$.

Example 2.3.6. Solve

$$\frac{dy}{dx} - 2xy = 2, \; y(0) = 1.$$

It is in normal form. So IF. is $\exp^{\int^x (-2x)}$. Mutiplying

$$e^{-x^2} \frac{dy}{dx} - 2xe^{-x^2} y = 2e^{-x^2} \Rightarrow \frac{d}{dx}[e^{-x^2} y] = 2e^{-x^2}.$$

$$[e^{-x^2} y]|_0^x = 2 \int_0^x e^{-t^2} dt \Rightarrow e^{-x^2} y - y(0) = 2 \int_0^x e^{-t^2} dt$$

Hence

$$y = e^{x^2} + 2e^{x^2} \int_0^x e^{-t^2} dt$$

$$y = e^{x^2} [1 + 2\sqrt{\pi} erf(x)]$$

2.4 Exact Differential Equation

Recall: If $f(x, y)$ is a differentiable function of two variables, then the **differential** of f is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Given the following form of 1-st order DE:

$$\boxed{M(x, y)dx + N(x, y)dy = 0}. \tag{2.15}$$

If this is an exact differential of some function, then it is called an **exact differential equation**. In other words, there exists a function $u(x, y)$ s.t.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = M dx + N dy = 0.$$

Hence the solution of the DE. (2.15) is $u(x, y) = c$. Thus, we must have

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N \tag{2.16}$$

and differentiation of M w.r.t y and differentiation of N w.r.t x gives

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

If $u \in C^2$ then by changing the order we must have

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.} \tag{2.17}$$

This is a necessary condition (in fact, it is sufficient also for most cases) for the DE (2.15) to be exact.

Integrating (2.16) gives

$$\boxed{u = \int M(x, y)\, dx + C_1(y)}$$

or

$$\boxed{u = \int N(x,y)\,dy + C_2(x)}.$$

Plug either of them into the D.E. and find $C_1(y)$ or $C_2(x)$. Here the relation $u(x,y) = C$ gives an implicit form of solution $y(x)$.

Theorem 2.4.1. *Assume $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous on a rectangle region $a < x < b, c < y < d$. Then necessary and sufficient condition for the DE.*

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

to be exact is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}.$$

Example 2.4.2. Solve

$$(3x^2y^2 + y + \cos x)dx + (2x^3y + x)dy = 0.$$

Sol. Let

$$M = 3x^2y^2 + y + \cos x, \quad N = 2x^3y + x.$$

$$\frac{\partial M}{\partial y} = 6x^2y + 1, \quad \frac{\partial N}{\partial x} = 6x^2y + 1.$$

So this is exact. Hence

$$u = \int M\,dx = \int (3x^2y^2 + y + \cos x)dx = x^3y^2 + xy + \sin x + g(y).$$

Here, $g(y)$ is a function of y only. Differentiate w.r.t y we have

$$\frac{\partial u}{\partial y} = 2x^3y + x + g'(y).$$

Since this must coincide with N, we have $g'(y) = 0$, $g(y) = C$. Thus the solution is $u = x^3y^2 + xy + \sin x = C$.

□

Example 2.4.3. Solve the DE.

$$e^{\sin x}(1 + (x+y)\cos x)dx + e^{\sin x}dy = 0.$$

Sol. Let $M = e^{\sin x}(1 + (x + y)\cos x)$, $N = e^{\sin x}$. Then it is exact since $M_y = N_x$. So

$$u(x, y) = \int N(x, y)dy = ye^{\sin x} + C_1(x).$$

Differentiate w.r.t. x, we see that

$$y\cos x e^{\sin x} + C_1'(x) = M = e^{\sin x}(1 + (x + y)\cos x)$$

holds. So

$$C_1'(x) = e^{\sin x}(1 + x\cos x).$$

Hence

$$C_1(x) = xe^{\sin x} + C_2$$
$$u(x, y) = (x + y)e^{\sin x} + C_2.$$

2.4.1 Integrating factor- Nonexact made exact

What to do when the DE. is not exact.

$$\boxed{M(x, y)dx + N(x, y)dy = 0}. \tag{2.18}$$

Can we make it exact? The answer is sometimes 'yes'. Multiplying $\mu(x, y)$ to (2.18) we get

$$\boxed{\mu M dx + \mu N dy = 0}. \tag{2.19}$$

Assume this is exact, then we must have

$$\boxed{(\mu M)_y = (\mu N)_x}. \tag{2.20}$$

We try to find such a function $\mu(x, y)$. Assume for simplicity μ is a function of x only. Then

$$\mu M_y = \mu_x N + \mu N_x$$

from which we obtain

$$\boxed{\frac{d\mu(x)}{dx} = \frac{(M_y - N_x)}{N}\mu(x)}. \tag{2.21}$$

Question: Is this solvable?

If the quotient $\frac{(M_y - N_x)}{N}$ is again a function of x only, then we can find $\mu(x)$. The IF. in this case is

$$\mu(x) = e^{\int \frac{(M_y - N_x)}{N}dx}.$$

Similarly, when μ is a function of y only we have some chance of finding μ. So if

$$\mu_y M + \mu M_y = \mu N_x$$

we obtain

$$\boxed{\frac{d\mu(y)}{dy} = \frac{(N_x - M_y)}{M}\mu(y)}. \tag{2.22}$$

If the quotient $\frac{(N_x - M_y)}{M}$ is again a function of y only, then we can find $\mu(y)$.

The IF. in this case is

$$\mu(y) = e^{\int \frac{(N_x - M_y)}{M} dy}.$$

Example 2.4.4. The D.E. $xydx + (2x^2 + 3y^2 - 20)dy = 0$ is not exact. With $M = xy,\ N = 2x^2 + 3y^2 - 20$, we find

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}.$$

But

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y}$$

is a function of y only. Thus IF is $e^{\int \frac{3}{y} dy} = y^3$. So multiplying by y^3 the following is exact.

$$xy^4dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0.$$

Answer is $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C$.

2.5 Solution by substitutions

Homogeneous Equation

A function $M(x, y)$ is said to be **homogeneous equation** if the total degrees are the same in of all terms. Formally, $M(x, y)$ is homogeneous if

$$M(tx, ty) = t^n M(x, y)$$

for some n.

For example $x^3 + x^2y + y^3$ or $x^2/y^2 + x/y$ are homogeneous, but $x^2 + y/x$ is not.

Similarly, if $M(x, y)$ and $N(x, y)$ are homogeneous, the following type of DE

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **homogeneous differential equation**. Use the substitution $\boxed{y = ux}$ and $dy = xdu + udx$ to get

$$M(x, ux)dx + N(x, ux)(xdu + udx) = x^n[M(1, u)dx + N(1, u)(xdu + udx)] = 0.$$

Thus

$$[M(1, u) + N(1, u)u]dx + N(1, u)xdu = 0 \ or \ \frac{dx}{x} + \frac{N(1, u)du}{M(1, u) + N(1, u)u} = 0.$$

Another view: Homogeneous equation of degree n is can be also written as

$$\boxed{y' = f\left(\frac{y}{x}\right)}. \tag{2.23}$$

Use the substitution $\boxed{y = ux}$ we see $y' = u'x + u$ and hence

$$u'x + u = f(u).$$

Thus

$$\frac{du}{dx} = \frac{f(u) - u}{x}.$$

This is separable. So we can integrate it as

$$\boxed{\int \frac{du}{f(u) - u} = \int \frac{dx}{x}}. \tag{2.24}$$

Example 2.5.1. Solve

$$y' = \frac{y}{x} + \frac{x}{y}, \quad y(1) = 2.$$

Sol. Let $y = ux$, $y' = u'x + u$ so that

$$u'x + u = u + \frac{1}{u}.$$

$$uu' = \frac{1}{x}.$$

Integrate

$$\frac{u^2}{2} = \ln |x| + C.$$

With IC., we see

$$y = x\sqrt{2 \ln x + 4}.$$

$\lrcorner$

Example 2.5.2.

$$y' = \frac{x^2 + 3xy}{x^2}.$$

Sol. $y = ux$, $f(u) = 1 + 3u$ Solution is

$$\int \frac{du}{1 + 3u - u} = \int \frac{dx}{x}.$$

$$\ln(1 + 2u) = 2 \ln x + C$$

and

$$1 + 2u = cx^2, \quad u = \frac{cx^2 - 1}{2}.$$

So

$$y = \frac{cx^3 - x}{2}.$$

□

Example 2.5.3. Solve

$$(x^2 + y^2)dx + (x^2 - xy)dy = 0.$$

Bernoulli equation

For real n the following DE.

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called the **Bernoulli equation**. For $n = 0, 1$ it is linear.

Assume $n \neq 0, 1$. Divide by y^n

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = f(x). \tag{2.25}$$

Use $w = y^{1-n}$ we get

$$\frac{dw}{dx} = (1 - n)y^{-n}\frac{dy}{dx}.$$

Subst. into (2.25) we get

$$\frac{dw}{dx} + (1 - n)P(x)w = (1 - n)f(x).$$

Example 2.5.4. Solve

$$x\frac{dy}{dx} + y = x^2y^2.$$

Sol. Divide by x to get

$$y' + \frac{y}{x} = xy^2.$$

This is Bernoulli equation with $n = 2$, $P(x) = \frac{1}{x}$. Set $w = y^{-1}$ we obtain

$$\frac{dw}{dx} - \frac{w}{x} = -x.$$

The I.F. is $e^{-\int \frac{1}{t}dt} = \frac{1}{x}$. So

$$\frac{w'}{x} - \frac{w}{x^2} = -1, \quad \left(\frac{w}{x}\right)' = -1.$$

$$\frac{w}{x} = -x + c, \quad y = \frac{1}{-x^2 + cx}.$$

□

Reduction to separable equation

Solve $\frac{dy}{dx} = (-2x + y)^2 + 7, \ y(0) = 0.$

Sol. Let $u = -2x + y$, the $du/dx = -2 + dy/dx$. So

$$\frac{du}{dx} + 2 = u^2 - 7, \ y(0) = 0 \ or \ \frac{du}{dx} = u^2 - 9.$$

This is separable.

$$\frac{du}{(u-3)(u+3)} = dx.$$

Exercise 2.5.5. (1) Find the solution.

(a) $(x^2 - 2xy + 1)dx + (3y^2 - x^2 + 2)dy = 0$

(b) $(xy^2 + y)dx + (x^2 + x)dy = 0$

(c) $(e^x \sin y - 2y \sin x)dx + (e^x \cos y + 2\cos x)dy = 0$

(d) $(\frac{y}{x} + 2)dx + (\ln x + 1)dy = 0$

(e) $(9x^2 + y - 1)dx + (4y + x)dy = 0$

(f) $y^2dx + 2xydy = 0$

(g) $(x + 1)e^x - e^y - xe^y\frac{dy}{dx} = 0$

(h) $3x^2 + 2yy' = 0$

(i) $3x^2(y+2)^2dx + 2x^3(y+1)dy = 0$

(j) $(e^x \cos y + x + y)dx + (-e^x \sin y + x + y)dy = 0$

(k) $\frac{dy}{x} - \frac{y}{x^2}dx = 0$

(l) $(\tan y + 3x^2)dx + x\sec^2 y dy = 0$

(2) Given the following DE, answer the following question.

$$(y^2 + 2xy)dx - x^2dy = 0$$

(a) Show this is not exact.

(b) Multiply y^{-2} and show the resulting equation is exact and find the solution.

(c) Is there any other solution?

(3) Given the following DE, answer the following question.

$$(5x^2y + 6x^3y^2 + 4xy^2)dx + (2x^3 + 3x^4y + 3x^2y)dy = 0$$

(a) Show this is not exact.

(b) Multiply $x^m y^n$ and find m, n so that the resulting equation is exact and find the solution.

Exercise 2.5.6. (1) The following is either homog. type or easily changed to homog. type. Solve them.

(a) $y' = \frac{y-x}{y-2x}$

(b) $y' = \frac{4y-3x}{x-y}$

(c) $y' = \frac{y-x-4}{y+x-2}(x = u-k, y = v-h)$

(d) $y' = \frac{x+3y-5}{x+y-1}$

(e) $y' = \frac{3xy+y^2}{x^2-xy}$

(f) $y' = \frac{y^2+2xy}{x^2}$

(g) $y' = \frac{x^2+xy+y^2}{x^2}$

(h) $y' = \frac{x+3y}{x-y}$

(i) $y' = \frac{y-x+5}{2x-y+4}$

(j) $y' = \frac{y}{x} + \frac{x^2\cos x}{y}(u = y/x)$

(k) $xyy' = y^2 - x^2$

(l) $x^2y' = x^2 - xy + y^2$

(m) $xyy' = x^2 + y^2$

(n) $xy' - y = x^2e^{\frac{y}{x}}$

(o) $xy' = y + \frac{x^3e^x}{y}$

(2) Use $y = ux^2$ solve $y' = \frac{2y}{x} + x\cos\left(\frac{y}{x^2}\right)$.

(3) Solve

$$\frac{dx}{dt} = -axy, \quad \frac{dy}{dt} = -bx$$

If $x(0) = x_0, \quad y(0) = y_0$ show that

$$ay^2 - 2bx = ay_0^2 - 2bx_0$$

2.6 A Numerical Method

2.7 Linear Model

Example 2.7.1 (Bacteria Growth)**.** Initially P_0. After 1 min. the number is $\frac{3}{2}P_0$. Find the equation.

$$\frac{dP}{dt} = kP, \ P = Ce^{kt}.$$

$$\frac{3}{2}P_0 = P_0e^k \Rightarrow k = \ln\frac{3}{2} = 0.4055 \Rightarrow P(t) = P_0e^{0.4055t}.$$

Example 2.7.2 (half life of plutonium 239)**.**

$$\frac{dA}{dt} = kA, \ A = Ce^{kt}.$$

After 15 years, the plutonium decayed by 0.043%. Find half life.

$$A(15) = A_0e^{15k} = 0.999574A_0 \Rightarrow k = \frac{1}{15}\ln 0.99957 = -0.00002867 \Rightarrow A(t) = A_0e^{-0.00002867t}.$$

$$\frac{1}{2}A_0 = A_0e^{-0.00002867t} \Rightarrow t = \frac{\ln 2}{0.00002867} = 24,180yrs.$$

Example 2.7.3. The D.E. $\frac{dP}{dt} = P(a - bP)$ has equilibrium points $P = 0$ and $P = a/b$. The behavior of solution near the Critical points are given in the table.

Many phenomena in the nature of engineering is expressed in terms of DE. The processes are called **(mathematical modelling)** If the position of free falling object is denoted by y(if we ignore the friction) then $y'' = g$, if the friction is considered we have $y'' = g - k(y')^2$

Example 2.7.4. [**Population of dynamics, Decay of isotope**] The mass of an isotope was 5 gram at a moment. After 3 min, it became 4 gram. Then let $y(t)$ be the remaining mass at t min. Write an equation of $y(t)$. What is half life ?(The time that takes to reduce to half of the original amount.)

Sol. $y(t)$ The decay rate $\frac{dy}{dt}$ is proportional to the current amount.

$$\frac{dy}{dt} = -ky \quad (k > 0).$$

Here k is some constant dependent on the material. The solution is $y(t) = Ce^{-kt}$. To find C we use initial condition $y(0) = 5$ we obtain

$$y(t) = 5e^{-kt}.$$

After 3 min. we see

$$4 = 5e^{-3k}.$$

Hence $k = -\frac{1}{3}\ln\frac{4}{5}$ and $y(t) = 5e^{\frac{t}{3}\ln\frac{4}{5}}$. Let the half life be denoted by t_0, then

$$y(t_0) = 5e^{\frac{t_0}{3}\ln\frac{4}{5}} = \frac{5}{2}.$$

Hence half life is $t_0 = 3\ln\frac{1}{2}/\ln\frac{4}{5}$.

□

Example 2.7.5. [Newton's law of cooling]

$$\frac{dT}{dt} = k(T - T_m).$$

2.8 Nonlinear Model

Example 2.8.1. [Spread of disease]

$$\frac{dx}{dt} = kxy$$

or

$$\frac{dx}{dt} = kx(n + 1 - x).$$

Example 2.8.2. [Oscillating pendulum] A pendulum of mass m is attached to a string of length ℓ. Let θ be the angle with vertical line. See figure 2.6 Find equation of θ.

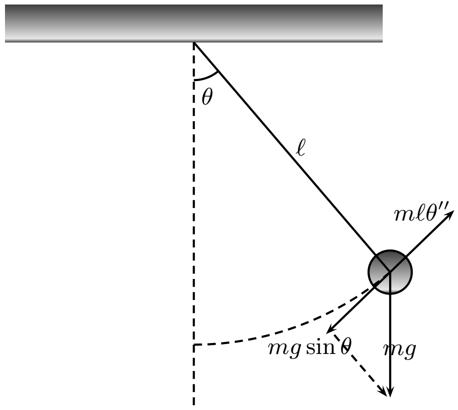

Figure 2.6: pendulum

Sol. The length of path of the pendulum is $\ell\theta$. Acceleration along the path is $\ell\theta''$. The force is by Newton's law $m\ell\theta''$. Gravitational force acting tangentially is $mg\sin\theta$. Hence

$$m\ell\frac{d^2\theta}{dt^2} + mg\sin\theta = 0 \quad \text{or} \quad \frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0.$$

This is nonlinear. When θ is small we use $\sin\theta \sim \theta$ to have

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0.$$

And its solution is

$$\theta = C_1 \cos\sqrt{\frac{g}{\ell}}t + C_2 \sin\sqrt{\frac{g}{\ell}}t.$$

□

Example 2.8.3. [Draining a Tank] Torricelli's law: The speed of outflux of water through a sharp edged hole at the bottom of a tank filled to a depth of h is the same as the speed of the body would require in the free fall from the height h: Thus,

$$\frac{dV}{dt} = -A_h\sqrt{2gh}$$

where A_h is the area of hole and h the height of water.

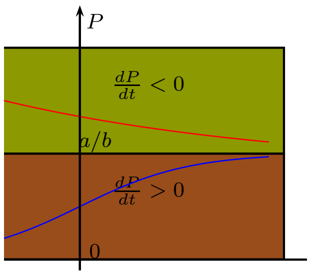

Figure 2.7: Logistic Curve

Example 2.8.4. [Logistic Equation] This is a particular form of a population model. It is given as

$$\frac{dP}{dt} = P(r - \frac{r}{K}P) \equiv P(a - bP). \tag{2.26}$$

Sol. Separation of variables. Write it as

$$\frac{dP}{P(a-bP)} = dt.$$

$$\left(\frac{1/a}{P} + \frac{b/a}{a-bP}\right) dP = dt.$$

$$\begin{aligned} \frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| &= t + c \\ \ln\left|\frac{P}{a-bP}\right| &= at + ac \\ \frac{P}{a-bP} &= c_1 e^{at}. \end{aligned}$$

Hence its solution is

$$P(t) = \frac{ac_1 e^{at}}{1 + bc_1 e^{at}}.$$

Use I.C $P(0) = P_0$ to get c_1.

$\lrcorner$

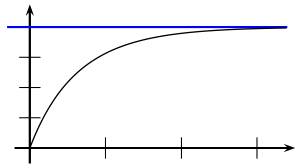

Figure 2.8: Chemical compound

Example 2.8.5. [Chemical reaction]

$$\frac{dX}{dt} = \left(a - \frac{M}{M+N}X\right)\left(b - \frac{N}{M+N}X\right) = k(\alpha - X)(\beta - X). \tag{2.27}$$

$$\frac{dX}{dt} = k\,(250 - X)\,(40 - X)$$

Use separation of var.

$$-\frac{1/210}{250 - X}dX + \frac{1/210}{40 - X}dX = kdt.$$

Integrating,

$$\ln\left|\frac{250 - X}{40 - X}\right| = 210kt + C, \quad \frac{250 - X}{40 - X} = C_2 e^{210kt}.$$

Use the conditions $X(0) = 0$, $X(10) = 30$ we can get k and C_2 and

$$X(t) = 1000\frac{1 - e^{-0.1258t}}{25 - 4e^{-0.1258t}}.$$

2.9 Modeling with systems of first order DEs

Systems of DE

Consider two(or more) unknown functions $x(t)$ and $y(t)$ satisfying

$$\begin{aligned} \tfrac{dx}{dt} &= g_1(t, x, y) \\ \tfrac{dy}{dt} &= g_2(t, x, y). \end{aligned} \tag{2.28}$$

A Predator-Prey Model

Two different species of animals compete to survive with the same environment. Suppose the first species(Prey) eats only vegetable, but the second one(Predator) eats first species only.

For examples, wolves eat caribou(deers), sharks devour little fish, fox eats rabbits. Let $x(t)$ denote the number of foxes and $y(t)$ denote the number of rabbits.

$$\begin{aligned} \frac{dx}{dt} &= -ax + bxy = x(-a + by) \\ \frac{dy}{dt} &= dy - cxy = y(d - cy). \end{aligned} \tag{2.29}$$

This is called **Lotka-Volterra predator-prey model**.

Competition Model

Two different species of animals compete to survive with the same environment. But this time, nor Prey-Predator relation, but they eats the same resources(such as food and space)

In absence of the other, the rate which each population grows is

$$\frac{dx}{dt} = ax \text{ and } \frac{dy}{dt} = cy. \tag{2.30}$$

However, with the existence of the other species they have to compete for food and spaces etc. So the equations becomes

$$\begin{aligned} \frac{dx}{dt} &= ax - by \\ \frac{dy}{dt} &= cy - dx. \end{aligned} \tag{2.31}$$

Networks - RLC Circuit

Kirchhoff's Voltage Law

Theorem 2.9.1 (Kirchhoff's Voltage Law)**. The sum of all voltage drop in the closed circuit is zero.**

Example 2.9.2. [Electric circuit] If the charge of the condenser is $Q(t)$, input voltage is $E(t)$, current I, R the voltage drop across the resistor is E_R, then by Ohm's law we have

$$E_R = RI. \tag{2.32}$$

Let the voltage drop across the coil be E_L. Then

$$E_L = L\frac{dI}{dt}. \tag{2.33}$$

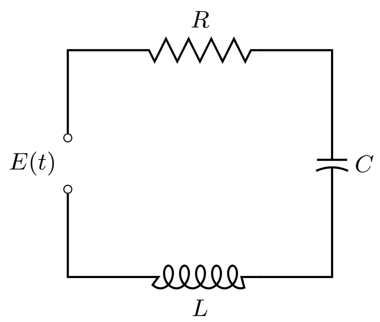

Figure 2.9: RLC-circuit

Also, with the charge in the condenser Q, the voltage drop across the condenser(capacitor) E_C satisfies

$$E_C = \frac{1}{C}Q.$$

Since the current is proportional to the rate of change of charge, we have $\frac{dQ}{dt} = I(t)$, thus,

$$E_C = \frac{1}{C}\int^t I(s)ds. \tag{2.34}$$

Example 2.9.3. [RLC**-circuit**] The input voltage is $E(t)$ the sum of (2.33), (2.35), (2.35) must be equal to $E(t)$. So we have

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(s)ds = E(t)$$

Taking derivative we get

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t). \tag{2.35}$$

Now find the solution with $E(t) = E_0 \sin \omega t$.

Sol. First we need to find a particular solution. Let

$$I_p(t) = A\cos\omega t + B\sin\omega t \tag{2.36}$$

and subt. it into equation (2.36) to get

$$A = \frac{-E_0 S}{R^2+S^2}, \quad B = \frac{E_0 R}{R^2+S^2}. \tag{2.37}$$

Here $S = \omega L - \frac{1}{\omega C}$ is the reactance. Simplify it

$$I_p(t) = I_0 \sin(\omega t - \delta). \tag{2.38}$$

Here

$$I_0 = \sqrt{A^2+B^2} = \frac{E_0}{\sqrt{R^2+S^2}}, \quad \tan\delta = \frac{S}{R}$$

and $\sqrt{R^2+S^2}$ is called impedance. Now need general solution of homg. eq. With two solutions λ_1, λ_2 of the char. eq.

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

we have

$$I_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Thus

$$I(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + I_0\sin(\omega t - \delta).$$

□

Interpretation: We see

$$\lambda_1 = \frac{-R + \sqrt{R^2 - \frac{4L}{C}}}{2L}, \quad \lambda_2 = \frac{-R - \sqrt{R^2 - \frac{4L}{C}}}{2L}.$$

Since the real part of λ_1, λ_2 are both negative, as time passed $I_h(t)$ approaches 0 and $I(t)$ approaches $I_p(t)$. $I_h(t)$ is called the **transient solution**, $I_p(t)$ is called the **steady-state solution**.

Example 2.9.4. $R = 10$(ohms), $L = 1$(henry), $C = 100^{-1}$(farad) RLC- $E(t) = -\cos 20t$. $I(0) = 0,\ I'(0) = 0$ $I(t)$.

Sol. From (2.36) we see

$$I'' + 10I' + 100I = 20\sin 20t$$

and since

$$S = \omega L - \frac{1}{\omega C} = 20 - \frac{100}{20} = 15$$

we have

$$A = \frac{-20 \cdot 15}{10^2 + 15^2} = -\frac{12}{13}, \quad B = \frac{20 \cdot 10}{10^2 + 15^2} = \frac{8}{13}.$$

Hence

$$I_p(t) = -\frac{12}{13}\cos 20t + \frac{8}{13}\sin 20t$$

and from $\lambda^2 + 10\lambda + 100 = 0$, $\lambda = -5 \pm 5\sqrt{3}i$, the homog. solution is

$$I(t) = c_1 e^{-5t}\cos 5\sqrt{3}t + c_2 e^{-5t}\sin 5\sqrt{3}t - \frac{12}{13}\cos 20t + \frac{8}{13}\sin 20t.$$

Using the IC.,

$$\begin{aligned} I(0) &= c_1 - \tfrac{12}{13} = 0 \\ I'(0) &= -5c_1 + 5\sqrt{3}c_2 + \tfrac{160}{13} = 0 \end{aligned}$$

from which we obtain $c_1 = \frac{12}{13}$, $\quad c_2 = -\frac{20}{13\sqrt{3}}$.

□

Exercise 2.9.5. (1) (a) $L = 10, R = 20, C = 0.01, E = 10\sin 100t, I(0) = I'(0) = 0$

(b) $L = 0.1, R = 20, C = 0.001, E = 10, I(0) = I'(0) = 0$

(c) $L = 2, R = 10, C = 0.1, E = 15, I(0) = 10, I'(0) = 0$

(2) The displacement of spring attached to ceiling with friction accounted is given by

$$mu'' + \gamma u' + ku = F(t).$$

Find solution when

(a) $m = 10, \gamma = 0.01, k = 2.5, F(t) = 1, u(0) = u'(0) = 0$

(b) $m = 0.5, \gamma = 0.2, k = 10, F(t) = \sin t, u(0) = u'(0) = 0$

(c) $m = 0.1, \gamma = 1, k = 10, F(t) = 0, u(0) = 2, u'(0) = 0$